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A SIMPLE CONSTITUTIVE RELATIONSHIP FOR ISOTROPIC SOLIDS WITH STRESS-STATE TYPE DEPENDENT PROPERTIES

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Abstract —The general form of the relationships between two symmetric coaxial tensors of second rank in an isotropic medium is considered. The condition under which the simple tensor linear form of these relationships, which includes the Lode's angle, can be used is determined. The range of validity of the proposed relationships is studied by means of Drucker's stability postulate and the existence and uniqueness conditions of the generalised solution of the problems. The applications that are built on the proposed form of the constitutive relationships, that cover creep, plastic, and elastic behaviour are given briefly.

1. INTRODUCTION

Mechanical behaviour of some homogeneous, isotropic materials is known to depend on the stress-state type, e.g. tension and compression. The aforementioned dependence was observed in elastic and elastic–plastic strain range as well as in rates of steady-state creep and failure characteristics. A description of a variety of appropriate materials may be found in the papers listed in the references.

In developing the constitutive models suitable for these materials two approaches were used by workers, mainly in the former Soviet Union. According to Leonov *et al.* (1966), Bykov (1971), Panferov (1968), Lomakin (1980) Zolochevskiy (1985), the stress-state type is defined by means of the first invariant of the stress tensor while the other investigators (Matchenko and Tolokonnikov, 1968; Tolokonnikov, 1968; Tsvelodub, 1977; Gorev *et al.*, 1979; Kadashevich *et al.*, 1990) have introduced the Lode's angle, or the third invariant of the stress tensor. A description of the dependence of the material properties on the stress-state type based on the first invariant of the stress tensor leads to theories more suitable for the mechanics of soils. Besides, the constitutive relationships for an incompressible medium that includes the first invariant of the stress tensor do not satisfy the material stability postulate (Tsvelodub, 1978). Implementation of these relationships can lead to non-uniqueness of the solution of the boundary value problems of the theory under consideration. However, using the Lode's angle results in the tensor, non-linear constitutive relationships can be very cumbersome. The choice of which approach should be used depends on the properties of the material under consideration.

The objective of this paper is to present the conditions under which a simple tensor linear form of the constitutive relationships, which includes the Lode's angle, may be used. It is shown that under some restrictions for the value of the phase of similitude of the tensor's deviators a simple form of the constitutive relationships is acceptable. The range of validity of the proposed relationships is studied by means of Drucker's stability postulate and the conditions of the existence and uniqueness of the generalised solution of the problems. The conditions of proportional loading are determined. The applications, built on the proposed form of the constitutive relations are given briefly.

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2. A SIMPLE FORM OF THE CONSTITUTIVE RELATIONSHIPS

Consider two symmetric tensors of the second rank a_{kl} and b_{kl}, in the Cartesian coordinate system. As follows from the representation theorem the general form of the relationship between these tensors is

$$a_{kl} = c_0\delta_{kl} + c_1 b_{kl} + c_2 b_{kn} b_{nl} \quad (kl = 1, 2, 3,) \tag{1}$$

where δ_{kl} is the unit tensor and c_i are the scalar coefficients. Assuming the tensors a_{kl}, b_{kl} to be co-axial, Novozhilov (1961) presented eqn (1) in a form where the coefficients c_i are expressed as functions of invariants of the tensors a_{kl} and b_{kl}. Following Tsvelodub (1974) this relationship can be represented in the form

$$a_{kl} = 3A_1 \frac{\partial B_1}{\partial b_{kl}} + W^0 \left(\frac{1}{B_2} \frac{\partial B_2}{\partial b_{kl}} - \tan\omega \frac{\partial \beta}{\partial b_{kl}} \right), \quad (kl = 1, 2, 3), \tag{2}$$

where $a^0_{kl} = a_{kl} - \delta_{kl} A_1$, $b^0_{kl} = b_{kl} - \delta_{kl} B_1$ are the tensor deviators; $A_1 = \frac{1}{3} a_{kk}$, $B_1 = \frac{1}{3} b_{kk}$, $A_2 = (\frac{1}{3} a^0_{kl} a^0_{kl})^{1/2}$, $B_2 = (\frac{1}{3} b^0_{kl} b^0_{kl})^{1/2}$ are the invariants of the corresponding tensors; $\omega = \alpha - \beta$ is a phase of similitude of the deviators; α and β are the Lode's angles, such that $\cos 3\alpha = \sqrt{2} A_2^{-3} \det a^0_{kl}$ and $\cos 3\beta = \sqrt{2} B_2^{-3} \det b^0_{kl}$; $W^0 = a^0_{kl} b^0_{kl}$ is the mixed invariant.

It is easy to show that the tensors with components $\partial B_1/\partial b_{kl}$, $\partial B_2/\partial b_{kl}$, $\partial \beta/\partial b_{kl}$ are orthogonal to each other with respect to the suitable scalar product, hence they form a system of base tensors for a three-dimensional space of the coaxial symmetric tensors of second rank. The invariants B_1, B_2, β in eqn (2) can be treated as independent potentials. In general a single potential function for the tensor a_{kl} does not exist and the assumption of existence of the potential function severely limits the class of possible relationships between tensors a_{kl} and b_{kl}. Multiplying eqn (2) by db_{kl} and contracting we can find the conditions of existence for a potential function (Tsvelodub, 1974)

$$3\frac{\partial A_1}{\partial B_2} = \frac{1}{B_2}\frac{\partial W^0}{\partial B_1}; \quad 3\frac{\partial A_1}{\partial \beta} = -\frac{\partial}{\partial B_1}(W^0 \tan\omega); \quad \frac{1}{B_2}\frac{\partial W^0}{\partial \beta} = -\frac{\partial}{\partial B_2}(W^0 \tan\omega) \tag{3}$$

that does not always hold. In particular, the conditions (3) are satisfied if a function $\Phi(B_1, B_2, \beta)$ exists such that

$$A_1 = \frac{1}{3}\frac{\partial \Phi}{\partial B_1}; \quad W^0 = B_2 \frac{\partial \Phi}{\partial B_2}; \quad \tan\omega = -\frac{\dfrac{\partial \Phi}{\partial \beta}}{B_2 \dfrac{\partial \Phi}{\partial B_2}}. \tag{4}$$

Then we can write eqn (2) in the form

$$a_{kl} = \frac{\partial \Phi(B_1, B_2, \beta)}{\partial b_{kl}}, \quad (kl = 1, 2, 3). \tag{5}$$

The relationship (2) follows from the assumption of coaxiallity of two symmetric second rank tensors. In order to make the relationship between these tensors completely definite it is necessary to specify the functions $A_1(B_1, B_2, \beta)$, $W^0(B_1, B_2, \beta)$ and $\omega(B_1, B_2, \beta)$ which are related to the physical nature of the medium under consideration. Due to the phenomenological approach the most general form (2) can be simplified by means of valid assumptions, which are dependent on the nature of the material that falls within the model. Let us consider in detail the assumption that leads to the tensor linear relationships.

In general, the relationship (1) between two second rank tensors is non-linear. In the expression (2) $\partial\beta/\partial b_{kl}$ contains the non-linear term. In applications, only those variants of theory that lead to tensor linear relationships are practically used. Consider the general formulation of a theory of this kind.

Assume the relationships between the components a_{kl} and b_{kl} as

$$a_{kl} = c_0\delta_{kl} + c_1 b_{kl}, \tag{6}$$

i.e. tensor a_{kl} depends linearly on tensor b_{kl}, although, due to the invariant coefficients c_i, the components b_{kl} enter into eqn (6) in a non-linear manner. Referring to eqn (2) this form of the relationships can be considered as following from the assumption $\omega(B_1, B_2, \beta) \equiv 0$ (similitude assumption). In order to describe the dependence of material properties on invariant β let us specify the relationship (6) as

$$a_{kl} = 3A_1(B_1, B_2, \beta)\frac{\partial B_1}{\partial b_{kl}} + W^{(t)}(B_1, B_2, \beta)\frac{1}{B_2}\frac{\partial B_2}{\partial b_{kl}}, \quad (kl = 1, 2, 3). \tag{7}$$

That is, the functions $A_1(B_1, B_2, \beta)$ and $W^{(t)}(B_1, B_2, \beta)$ depend on the value β, and tensor a_{kl} depends linearly on tensor b_{kl}.

Let us consider the geometrical interpretation of the reductions, leading from eqns (2)–(7). In order to simplify the observation, we introduce the assumptions $A_1(B_1, B_2, \beta) \equiv 0$ (incompressibility assumption) and $\omega(B_1, B_2, \beta) = \omega(\beta)$, i.e. the phase of similitude of deviators depends solely on the angle β. In other words, the invariant α of the tensor a_{kl} is a function of the corresponding invariant β of the tensor b_{kl} that includes the classical assumption $\alpha = \beta$, i.e. $\omega(B_1, B_2, \beta) \equiv 0$. Following the assumptions, relationship (2) is reduced to (Tsvelodub, 1974)

$$a_{kl} = W^{(t)}(\Sigma_1)\frac{1}{\Sigma}\frac{\partial \Sigma}{\partial b_{kl}}; \quad \Sigma = B_2 \exp\gamma(\beta); \quad \gamma(\beta) = -\int_{\beta_0}^{\beta} \tan\omega \, d\beta. \tag{8}$$

From eqn (8) it follows that if $\omega(B_1, B_2, \beta) \equiv 0$, then $\Sigma = B_2$, and hence the tensor linear relationship (7) is reduced to

$$a_{kl} = W^{(t)}(\Sigma_1)\frac{1}{B_2}\frac{\partial B_2}{\partial b_{kl}}, \quad (kl = 1, 2, 3). \tag{9}$$

Therefore, the vector a_{kl} defined by eqn (2) is directed along the normal to the surface $\Sigma = \text{const.}$, while eqn (7) defines the vector that is directed along the normal to surface $B_2 = \text{const.}$ Since $(1/B_2)(\partial B_2/\partial b_{kl})$ is of the same order as $\partial\beta/\partial b_{kl}$, from eqn (2) it follows that the condition of proximity of these surfaces is $\|\tan\omega(B_1, B_2, \beta)\| \ll 1$ where $\|\cdot\|$ denotes the norm of the function. Note, that in general $\Sigma \neq \Sigma_1$, since the surface $\Sigma = \text{const.}$ defines the direction of the vector a_{kl}, while the surface $\Sigma_1 = \text{const.}$ defines the equal intensity processes if the function W_0 is taken as the measure of intensity of the process. Hence, the particular tensor linear form (7) as well as the general form (2) defines the nonassociated rule.

Consider from the same point of view the relationships (5), when a potential function $\Phi(B_1, B_2, \beta)$ is assumed to exist. The tensor linear relationship can be deduced from (5) by using, as before, the assumption $\omega(B_1, B_2, \beta) \equiv 0$; but, by virtue of this assumption, from eqn (4) it follows that $\Phi(B_1, B_2, \beta)$ can not depend on the invariant β. However, for prescribed $kl = 1, 2, 3$ the functions defined by eqns (2) and (7) are asymptotically equal as ω approaches zero. We can then consider the relationship

$$a_{kl} = \frac{1}{3}\frac{\partial \Phi}{\partial B_1}\delta_{kl} + \frac{1}{3B_2}\frac{\partial \Phi}{\partial B_2}b_{kl}, \quad (kl = 1, 2, 3), \tag{10}$$

where the function $\Phi(B_1, B_2, \beta)$ depends on the invariant β, as resulting from an approximation of eqn (5). It is evident that the function $\Phi(B_1, B_2, \beta)$, as well as any other function of the invariant β, is not a potential function for the tensor linear relationship (10). The condition of proximity for the functions defined by eqns (5) and (10) is $\|\tan\omega(B_1, B_2, \beta)\| \ll 1$ which implies a restriction for the function $\omega(B_1, B_2, \beta)$.

In general, the assumption of the existence of a potential function is not strictly necessary, however this assumption allows us to formulate the variational theorems that are useful for developing approximate methods. The non-conventional variational principle, (Turovtsev, 1988), based on the local potential concept (Glansdorff and Prigogine, 1954) was proposed as suitable criterion describing the stationary state of the continua defined by the non-associated rule (2).

The tensor linear form (7) of the general relationship (2) can be used in order to describe the behaviour of the solids with complex properties, when non-associativeness of the constitutive rule is acceptable. When the existence of a potential function is assumed or follows from a reliable physical principle, the relationship (10) can be treated as a reasonably good approximation for the tensor non-linear relationship (5), as good an approximation as the value of $\|\tan\omega(B_1, B_2, \beta)\|$ is small in comparison with unity. This relationship takes into account the dependence of the material properties of the Lode's angle and can be easily used in applications.

3. STABILITY CONDITIONS

In order to completely determine the relationships between the tensors a_{kl} and b_{kl} it is necessary to define the functions $A_1(B_1, B_2, \beta)$, $W^0(B_1, B_2, \beta)$ and $\omega(B_1, B_2, \beta)$ or only $\Phi(B_1, B_2, \beta)$. These functions may assume various forms which, can not be completely arbitrary. More specific formulation can be obtained if one takes into consideration the Thermodynamics Laws, but thermodynamic relationships are strictly applicable only to reversible equilibrium processes. For the relationships (2), (7) and (10) there are no potential functions, therefore, when employing these theories, one may encounter circumstances in which certain consequences of the theory appear to be contradictory or physically unacceptable. We, therefore, require a simple criteria which must, as a minimum requirement, be satisfied by any theory. The "quasi-thermodynamic" Drucker's postulate of stability of material under isothermal conditions (Drucker, 1959) will be used for deducing the restrictions imposed on functions $A_1(B_1, B_2, \beta)$, W^0 (B_1, B_2, β), $\omega(B_1, B_2, \beta)$ and $\Phi(B_1, B_2, \beta)$, as well as for deducing the associated properties.

For arbitrarily small increments δa_{kl} of the tensor a_{kl} and the corresponding increments δb_{kl} of the tensor b_{kl}, the consequence of Drucker's stability postulate may be written as

$$\delta a_{kl}\delta b_{kl} \geqslant 0. \tag{11}$$

Consider this property as characterizing a certain stability of the material without referring to any mechanical meaning. Apply the inequality (11) in order to obtain the restrictions on the functions $A_1(B_1, B_2, \beta)$, $W^0(B_1, B_2, \beta)$, $\omega(B_1, B_2, \beta)$ and $\Phi(B_1, B_2, \beta)$. Assume that these functions are differentiable with respect to B_1, B_2 and β.

Choosing the coordinate axes to coincide with the principal axes of the tensor b_{kl}, the principal values of the tensor may be expressed as

$$b_1 = \sqrt{2B_2}\cos\beta + B_1, \quad b_2 = \sqrt{2B_2}\cos(\beta + \tfrac{4}{3}\pi) + B_1, \quad \text{and}$$
$$b_3 = \sqrt{2B_2}\cos(\beta + \tfrac{2}{3}\pi) + B_1,$$

with $b_1 \geqslant b_2 \geqslant b_3$, $0 \leqslant \beta \leqslant \pi/3$. Since the tensors a_{kl} and b_{kl} are assumed to be coaxial, the same relations are valid for tensor a_{kl} and one can obtain

$$\delta a_{kk}\delta b_{kk} = 3\delta A_1\delta B_1 + \delta W^0\left(\frac{\delta B_2}{B_2} - \tan\omega\delta\beta\right) - W^0\left(\left(\frac{\delta B_2}{B_2}\right)^2 + \frac{\delta\omega\delta\beta}{\cos^2\omega}\right) + W^0\left((\delta\beta)^2 + \frac{2\tan\omega}{B_2}\delta B_2\delta\beta\right) \quad (12)$$

using eqn (2) (Tsvelodub, 1978). Under the same assumptions we obtain:

$$\delta a_{kk}\delta b_{kk} = 3\delta A_1\delta B_1 + \delta W^0\frac{\delta B_2}{B_2} - W^0\left(\frac{\delta B_2}{B_2}\right)^2 + W^0(\delta\beta)^2 \quad (13)$$

for the particular form (7) of the tensor linear relationship.

Evaluating the variations δW^0, $\delta\omega$, and δA_1, corresponding to the respective variations δB_2, $\delta\beta$ and δB_1, we can reduce eqns (12) and (13) to quadratic forms in the variables δB_2, $\delta\beta$ and δB_1. Then from eqn (11) it follows that these quadratic forms must be non-negative.

For the case when the principal axes of the tensor b_{kl} do not coincide with the coordinate axes the quadratic form (12), as well as (13), is reduced to two quadratic forms: one is associated with the variations δB_2, $\delta\beta$, and δB_1 as above, and the other is associated with rotation of the principal axes of the tensor b_{kl}. It can be shown (Tsvelodub, 1978) that the quadratic form associated with rotation of the principle axes through any point in an isotropic medium is non-negative definite. Hence, for the class of stable isotropic material, when the tensor a_{kl} depends linearly on the tensor b_{kl}, we obtain

$$\delta a_{kl}\delta b_{kl} = q_{ij}x_ix_j \geqslant 0, \quad (14)$$

where $x_1 = B_2$, $x_2 = \beta$, $x_3 = B_1$ and q_{ij} are the elements of the symmetric matrix:

$$q_{11} = \frac{1}{B_2}\frac{\partial W^0}{\partial B_2} - \frac{W^0}{B_2^2}, \quad q_{22} = W^0, \quad q_{33} = 3\frac{\partial A_1}{\partial B_1},$$

$$q_{12} = \frac{1}{2B_2}\frac{\partial W^0}{\partial\beta}, \quad q_{13} = \frac{1}{2B_2}\frac{\partial W^0}{\partial B_1} + \frac{3}{2}\frac{\partial A_1}{\partial B_2}, \quad q_{23} = \frac{3}{2}\frac{\partial A_1}{\partial\beta}. \quad (15)$$

The equality sign in eqn (14) holds only if $\delta b_{kl} = 0$, $(kl = 1, 2, 3)$. Hence, we may employ the test for positive definiteness of the matrix $[q_{ij}]$:

$$q_{11} > 0, \quad q_{11}q_{22} - q_{12}^2 > 0, \quad \det[q_{kl}] > 0. \quad (16)$$

From eqn (16) one can obtain restrictions on the functions $A_1(B_1, B_2, \beta)$, $W^0(B_1, B_2, \beta)$, and $\omega(B_1, B_2, \beta)$, and on the derivatives of these functions.

In the case when the tensor linear relationship (10) is defined by a single function $\Phi(B_1, B_2, \beta)$ one can obtain restrictions on Φ and on derivatives of Φ by using the conditions (16). Substituting (4) into (15) we obtain the matrix $[q_{ij}]$ whose coefficients are in the form

$$q_{11} = \frac{\partial^2\Phi}{\partial B_2^2}, \quad q_{22} = B_2\frac{\partial\Phi}{\partial B_2}, \quad q_{33} = \frac{\partial^2\Phi}{\partial B_1^2},$$

$$q_{12} = \frac{1}{2}\frac{\partial^2\Phi}{\partial B_2\partial\beta}, \quad q_{13} = \frac{\partial^2\Phi}{\partial B_1\partial B_2}, \quad q_{23} = \frac{1}{2}\frac{\partial^2\Phi}{\partial\beta\partial B_1}. \quad (17)$$

The conditions (16), with coefficients defined by (15) and (17), are necessary and sufficient for eqn (11). Consequences of the postulate (11) have far-reaching implications in the theory developed as well as in the general case of relationship (2). The uniqueness of the solution can be deduced from these conditions (Tsvelodub, 1978), as well as the structure of the constitutive relationships of the theory.

Consider a stable medium when the relationship between the tensors a_{kl}, b_{kl} has the form (7) or (10). Note that assuming $A_1(B_1, B_2, \beta) = 0$, from (4) it follows that the function $\Phi(B_1, B_2, \beta)$ can not depend on the invariant B_1, although in the general case of relationship (7) the function $W^0(B_1, B_2, \beta)$ still depends on B_1. Following Tsvelodub (1978), we can deduce from conditions (16) that, owing to the stable medium described by the tensor linear relationship (7) was incompressible ($A_1 \equiv 0$) it is necessary and sufficient that the function $W^0(B_1, B_2, \beta)$ does not depend on invariant B_1.

Necessity. Let a medium be stable and incompressible ($A_1 \equiv 0$). Then all the principal minors of matrix $[q_{ij}]$ with coefficients (15) are non-negative, in particular

$$\begin{vmatrix} q_{11} & q_{13} \\ q_{31} & q_{33} \end{vmatrix} \geqslant 0, \quad \begin{vmatrix} q_{22} & q_{23} \\ q_{32} & q_{33} \end{vmatrix} \geqslant 0. \tag{18}$$

Since $q_{33} = 0$ by virtue of eqn (15), from (18) it follows that $q_{23} = q_{33} = 0$. Thus, with reference to (15) we obtain $\partial W^0/\partial B_1 = 0$.

Sufficiency. Let the medium be stable and the functions A_1, W^0, and ω are not dependent on the invariant B_1, then $\partial W^0/\partial B_1 = \partial\omega/\partial B_1 = \partial A_1/\partial B_1 = 0$. Hence $q_{33} = 0$ by virtue of eqn (15), and, as earlier, from eqn (18) it follows that $q_{23} = q_{33} = 0$. Referring again to eqn (15) we can deduce that $\partial A_1/\partial B_1 = \partial A_1/\partial B_2 = \partial A_1/\partial\beta = 0$, i.e. $A_1 =$ const., namely (Tsvelodub, 1978) $A_1(B_1, B_2, \beta) \equiv 0$. Hence, the constitutive relation for an incompressible medium described by relationship (7) is $W^0 = W^0(B_2, \beta)$.

The stability conditions (15) and (16) can be simplified for an incompressible medium as

$$a > 1, \quad a - \tfrac{1}{4}b^2 > 1, \quad a = \frac{B_2}{W^0}\frac{\partial W^0}{\partial B_2}, \quad b = \frac{1}{W^0}\frac{\partial W^0}{\partial\beta}. \tag{19}$$

Therefore, the constitutive relations for an incompressible medium ($A_1 \equiv 0$) that includes the invariant B_1 do not satisfy stability postulate (11). Implementation of these relations leads to non-uniqueness of the solution of the boundary value problems of the theory under consideration (Tsvelodub, 1978).

The conditions obtained define the class of stable materials whose behaviour can be described by the tensor linear relationships (7) and (10) and ensure uniqueness of the solution in dynamic as well as static problems.

4. CONDITIONS FOR PROPORTIONAL LOADING

In developing a deformation type theory of plasticity there is an important type of loading under which all the components of the tensor deviator are varied in proportion to one and the same parameter. This proportional loading is comparatively easy to achieve experimentally. Therefore, the laws of a theory of plasticity can be checked under these conditions.

Consider, following Il'yushin (1948), the conditions under which proportional loading can be maintained in a stable medium described by the tensor linear relationships (7) and (10). Assume that: (i) all the components of the tensor a_{kl} vary proportionally to the parameter $t \in (0, \infty)$, i.e. $a_{kl} = ta^*_{kl}$; (ii) $b_{kl} = \chi(t)b^*_{kl}$, where $\chi(t)$ is an unknown function of the parameter t and a^*_{kl}, b^*_{kl}, satisfies the equilibrium equations, the boundary conditions and compatibility conditions. Then tensors a_{kl}, b_{kl}, as well as a^*_{kl}, and b^*_{kl}, will be the solution of the equilibrium boundary value problem and only the constitutive equations must be satisfied. As was shown for an incompressible stable medium, the constitutive relationship is given by $W^0 = W^0(B_2, \beta)$, and by the use of relationship (7) we can obtain for this case

$$A_2 = \frac{1}{3B_2} W^0(B_2, \beta). \tag{20}$$

Assume $W^0(B_2, \beta)$ to be a homogeneous function of degree $n+1$ in the components of tensor b_{kl}: $W^0 = f(\beta)B_2^{n+1}$, where $f(\beta)$ is a function solely of angle β. Since angle β is a constant during proportional loading, by using the relations $A_2 = tA_2^*$, and $B_2 = \chi(t)B_2^*$ we can deduce that eqn (20) is satisfied if function $\chi(t)$ is of the following form: $\chi(t) = t^{1/(n+1)}$. Therefore, the sufficient condition for proportional loading of a stable, incompressible medium (7) during proportional change in the tensor a_{kl} components is the following relationship

$$A_2 = \tfrac{1}{3} f(\beta) B_2^{n-1}. \tag{21}$$

Note that condition (21) is only sufficient.

5. UNIQUENESS OF THE SOLUTION

As mentioned earlier, the material stability conditions (16) and (19) are sufficient for uniqueness of the solution. In Chircov (1988), conditions for existence and uniqueness of the generalised solution, in the sense of the theory of distributions, were obtained. A form of the relationship between the second invariants of the deviators of the tensors a_{kl} and b_{kl} the same as eqn (21), i.e. corresponding to the tensor linear relationship (7) was used. Although in Chircov (1988), the generalized formulation of the problem was used the existence and uniqueness conditions obtained turned out to be similar to the conditions (16).

As shown in Chircov (1988), if the deviators of the tensors a_{kl} and b_{kl} are proportional and coaxial, then a necessary and sufficient condition for the existence and uniqueness of the generalized solution of the problems is

$$\frac{\partial \bar{a}}{\partial \bar{b}} - \frac{\sqrt{3+\mu_b^2}}{\sqrt{3}\,\bar{b}} \left| \frac{\partial \bar{a}}{\partial \mu_b} \right| > 0, \tag{22}$$

where $\bar{a}$ $(\bar{b}, \mu_b)$ and $\bar{b}$, are the intensities of the tensors a_{kl} and b_{kl}, respectively, and μ_b is the Lode parameter of the tensor b_{kl}. Clearly, the assumption of proportionality of the tensor deviators a_{kl}^0 and b_{kl}^0 used in Chircov (1988) bound the admissible form of relationships $\bar{a} = \bar{a}(\bar{b}, \mu_b)$ by the tensor linear form of relationships (7) and (10).

By using the known relationships

$$\bar{a} = \frac{3}{\sqrt{2}} A_2, \quad \bar{b} = \sqrt{2B_2}, \quad \mu_b = -\sqrt{3}\tan(\beta), \tag{23}$$

we can represent the necessary and sufficient condition (22) as follows:

$$\left| \frac{\partial A_2}{\partial \beta} \right| < B_2 \frac{\partial A_2}{\partial B_2}. \tag{24}$$

Substituting (20) into (24) we obtain the desired restriction on function $W^0(B_2, \beta)$

$$\left|\frac{\partial W_0}{\partial \beta}\right| < B_2 \frac{\partial W_0}{\partial B_2} - W^0. \tag{25}$$

Consider, in particular, the case of $W^0 = f(\beta)B_2^{n+1}$. Then, from eqn (25) it follows that $|f'/f| < n$. The stability conditions (19) lead to $|f'/f| < 2\sqrt{n}$. Hence, if $n \geqslant 4$, the stability conditions hold, as well as the condition for existence and uniqueness of the generalized solution of the problems. If $n < 4$, then the stability conditions are broken, while the condition for existence and uniqueness of the generalized solution of the problems is held.

In Chircov (1988), by means of inequality (22) and the additional assumptions, it was proven that the convergence of the iterational method of the elastic solutions and the rate of convergence was estimated. Therefore, the condition (25) is a useful additional restriction in developing the constitutive relationships of the proposed theory.

6. APPLICATIONS

The particular forms of the relationships (7) and (10) studied here have already been presented in the literature. These models cover steady-state creep, plastic, and elastic behaviour.

The constitutive equations of steady-state creep for materials with different properties in tension and compression were developed by Rabotnov (1969), Gorev *et al.* (1979), and Tsvelodub (1974), in order to model the creep of light weight alloys. It was found experimentally that the creep rate depends on the type of stress state. Materials are incompressible and creep curves are similar for various types of the stress state. Besides, a negligible difference was observed between the surfaces $W^0(B_2, \beta) = \text{const.}$ under various values of the angle β. Tsvelodub, (1974) assumed that $W^0 = c_1(B_2 f(\beta))^{n+1}$ and $f(\beta) = (1 - c_2 \sin^2 3\beta)^{1/6}$. A non-associative flow rule was proposed

$$\dot{\varepsilon}^0_{kl} = W^0(B_2, \beta)\frac{1}{B_2}\frac{\partial B_2}{\partial \sigma_{kl}}, \quad (kl = 1, 2, 3), \tag{26}$$

where $W^0(B_2, \beta) = \dot{\varepsilon}^0_{kl}\sigma^0_{kl}$ is the dissipation function in creep, $\dot{\varepsilon}^0_{kl}$ is the deviator of the creep rate of deformation tensor, σ^0_{kl} is the stress tensor deviator. Using the stability condition (11) the estimation $-\frac{1}{2} \leqslant c_2 \leqslant \frac{1}{3}$ was obtained for the associated flow rule, while for the non-associated rule $-1 \leqslant c_2 \leqslant 2(\sqrt{2} - 1)$. Thus, the associated flow rule is restricted to a more limited class of materials. The computations carried out by using the tensor linear relations (26) proved to be in close agreement with the experimental results. The computational advantages of the proposed flow rule are evident. This model has been fully described by Tsvelodub (1974), and here a brief summary of the formulation is given.

It is well established in the literature that the plastic behaviour of some materials depends on Lode's angle. In order to describe the observed behaviour various forms of yield functions have been proposed, so that dependence on Lode's angle is included. If an isotropic material does not exhibit a significant degree of plastic dilation, then the general form of the yield function is

$$\Sigma(\tau, \varphi) = f(\varphi)\tau - \tau_s, \tag{27}$$

where $\tau = (\frac{1}{3}\sigma^0_{ij}\sigma^0_{ij})^{1/2}$ is the octahedral shear stress, τ_s is the octahedral yield stress, φ is the angle, related to Lode's parameter μ by eqn (23), and $f(\varphi)$ is the function which governs the shape of the yield surface in the octahedral plane. Various forms of the $f(\varphi)$ function were proposed. A typical form is $f(\varphi) = (1 - c\cos 3\varphi)^\alpha$, where the exponent α has been taken at various values from 1/3 to 1, and c is a parameter that is to be determined. In Tolokonnikov (1968) and Kadashevich and Pomytkin (1990) more complex continuous approximations of $f(\varphi)$ were considered, while in Danshin (1988), the piece-wise continuous forms were studied.

Associated with a yield function (27), the flow rule

$$\mathrm{d}\varepsilon_{kl}^{p} = \mathrm{d}\lambda \frac{\partial \Sigma}{\partial \sigma_{kl}} = \partial \lambda \left(\frac{\partial \Sigma}{\partial \tau} \frac{1}{3\tau} \sigma_{kl}^{0} + \frac{\partial \Sigma}{\partial \varphi} \frac{\partial \varphi}{\partial \sigma_{kl}} \right), \quad (kl = 1, 2, 3)$$

would be very cumbersome to use since $\partial\varphi/\partial\sigma_{kl}$ contains the tensor nonlinear term. Let us apply the above proposed analysis in order to simplify the plastic flow rule associated with yield function (27). Assume that tensors $\mathrm{d}\varepsilon_{kl}^{p}$ and σ_{kl}^{0} are coaxial, and $\omega = \omega(\varphi)$, then the general form of the relationships between $\mathrm{d}\varepsilon_{kl}^{p}$ and σ_{kl}^{0} is

$$\mathrm{d}\varepsilon_{kl}^{p} = \frac{\partial W^{p}(\Sigma)}{\bar{\Sigma}} \frac{\partial \bar{\Sigma}}{\partial \sigma_{kl}}, \quad (kl = 1, 2, 3), \tag{28}$$

where $\partial W^{p} = \sigma_{kl}^{0} \mathrm{d}\varepsilon_{kl}^{p}$ is the plastic dissipation, $\bar{\Sigma} = \bar{f}(\varphi)\tau$ is the plastic potential function, and Σ is some equivalent stress. Let us associate this equivalent stress with the yield function (27), that is, assume $\Sigma = f(\varphi)\tau$. In general $\Sigma \neq \bar{\Sigma}$, hence (28) defines a nonassociative flow rule, and in order to remove the dependence of the plastic potential $\bar{\Sigma}$ on the angle φ let us define the plastic potential as $\bar{\Sigma} = \tau$. The surface $\bar{\Sigma} =$ const. is then the von Mises cylinder with a circular yield locus. The yield function (27) is used as before and we obtain from (28) the nonassociative flow rule

$$\mathrm{d}\varepsilon_{kl}^{p} = \frac{\partial W^{p}(\Sigma)}{3\tau^{2}} \sigma_{kl}^{0} = \mathrm{d}\lambda \sigma_{kl}^{0}, \quad (kl = 1, 2, 3), \tag{29}$$

which is similar to the Levy–Mises relationships but $\mathrm{d}\lambda$ is a scalar factor that depends on the Lode's angle. By means of the model (27) and (29) it is possible to describe the different behaviour of a material in tension and compression in the plastic range of deformation. Under conditions (19) the constitutive relations (29) satisfy the stability postulate (14), incompressibility assumption, and are presented in a simple form. Various strain-hardening rules can be incorporated into the model in the usual way.

A similar approach to the development of constitutive relations was used by Reed and Cassie (1988). The Mohr–Coulomb yield criterion with the nonassociative flow rule provides a simple and useful model for geotechnical materials.

Many theories have been proposed in order to describe the elastic behaviour of materials with different properties in tension and compression. Among others we refer to the studies developed by Matchenko and Tolokonnikov (1968), Tolokonnikov (1968), Sarkisian (1971), Tsvelodub (1977) and Turovtsev (1981), where the parameter $\zeta = \cos 3\beta$ was used as the characteristic of the type of the stress state. Developing the theory proposed by Matchenko and Tolokonnikov (1968), Tsvelodub (1977) gives the following expression for the elastic potential function

$$\Phi(B_1, B_2, \zeta) = \Phi_1(B_1) + \Phi_2(f(\zeta)g(B_2)) = c_1(\operatorname{sign} B_1)B_1^2 + c_2(\zeta)g(B_2) \tag{30}$$

where $c_1(\operatorname{sign} B_1)$, $c_2(\zeta)$, and $g(B_2)$ are the material functions, and B_1, B_2, and $\zeta = \cos 3\beta$ are the corresponding invariants of the stress tensor. The prospect of finding partial derivatives of this function with respect to the stresses is a daunting one and the complexity of resulting relations seems unwarranted when we recall that a potential function (30) was proposed so that the dependence of the elastic constants c_1 and c_2 on the type of the stress state is included. Consider the alternative approach based on the approximate relationships (10). Let us assume for simplicity that

$$c_1(\operatorname{sign} B_1) \equiv 0, \quad g(B_2) = \frac{1}{n+1} B_2^{n+1}. \tag{31}$$

Then, by using (10) we obtain stress strain relations in the form

$$\varepsilon_{kl} = \tfrac{1}{3}c(\zeta)B_2^{n-1}\sigma^0_{kl}, \quad (kl = 1, 2, 3) \tag{32}$$

which combines computational simplicity in implementation with the possibility of describing the dependence of elastic properties on the type of stress state.

Substituting (31) into (30) we can obtain by means of (4), the condition of proximity of the tensor linear relationship (32) with the corresponding non-linear one in the form

$$|\tan\omega| = \left|\frac{3c_2'(\zeta)\sqrt{1-\zeta^2}}{(n+1)c_2(\zeta)}\right| \ll 1$$

from which it follows that

$$\left|\frac{c_2'(\zeta)}{c_2(\zeta)}\right| \ll \tfrac{1}{3}(n+1). \tag{33}$$

For the simple case a valid approximation of the function $c_2(\zeta)$ is $c_2(\zeta) = a\exp(c\zeta)$, where a and c are the material constants. Using this assumption in eqn (29) we deduce that the tensor linear relation (32) may be considered as an admissible approximation if $|c| \ll \frac{1}{3}(n+1)$. It is easy to show that under this restriction the stability conditions (16) are fulfilled.

Thus, the proposed stress strain rule (32) provides a simple and general model for the materials with stress-state type dependent properties.

7. CONCLUSIONS

The general form of the relationships between two coaxial symmetric tensors of the second order in an isotropic medium was studied. It has been shown that the particular tensor linear form of these relations can be used in order to describe the behaviour of solids with complex properties under some restriction for the value of the phase of similitude of the tensor's deviators. These relations should be considered as approximate when the associativeness of the flow rule is assumed or it follows from a reliable physical principle.

Conditions of material stability were obtained which ensure uniqueness of the solution in dynamic as well as static problems of the theory under consideration. These conditions have been compared with restrictions on the material functions, which follow from the conditions for existence and uniqueness of the generalised solution of the problems. The proposed constitutive relationships are shown to be in agreement with these restrictions.

Implementation of the tensor linear relations has been demonstrated on various constitutive models for stress-state type sensitive materials. These models cover steady-state creep, plastic, and elastic behaviour, and demonstrate the flexibility and simplicity of the approach.

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